

# ROUGH NORMS IN SPACES OF OPERATORS

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**ABSTRACT.** We investigate sufficient and necessary conditions for the space of bounded linear operators between two Banach spaces to be rough or average rough. Our main result is that  $\mathcal{L}(X, Y)$  is  $\delta$ -average rough whenever  $X^*$  is  $\delta$ -average rough and  $Y$  is alternatively octahedral. This allows us to give a unified improvement of two theorems by Becerra Guerrero, López-Pérez, and Rueda Zoca [J. Math. Anal. Appl. 427 (2015)].

## 1. INTRODUCTION

All Banach spaces considered in this paper are non-trivial and over the real field. The closed unit ball of a Banach space  $X$  is denoted by  $B_X$  and its unit sphere by  $S_X$ . The dual space of  $X$  is denoted by  $X^*$ , and the Banach space of all bounded linear operators acting from  $X$  to another Banach space  $Y$  by  $\mathcal{L}(X, Y)$ .

**Definition 1.1.** Let  $X$  be a Banach space and  $\delta > 0$ . The space  $X$  is said to be

- $\delta$ -rough [7] if, for every  $x \in S_X$ ,

$$\limsup_{\|y\| \rightarrow 0} \frac{\|x + y\| + \|x - y\| - 2}{\|y\|} \geq \delta;$$

- $\delta$ -average rough [3] if, whenever  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in S_X$ ,

$$\limsup_{\|y\| \rightarrow 0} \frac{1}{n} \sum_{i=1}^n \frac{\|x_i + y\| + \|x_i - y\| - 2}{\|y\|} \geq \delta.$$

The space  $X$  is said to be *non-rough*, if there is no  $\varepsilon > 0$  such that  $X$  is  $\varepsilon$ -rough.

A dual characterization of roughness is well known. The space  $X$  is  $\delta$ -rough if and only if the diameter of every weak\* slice of  $B_{X^*}$  is greater than or equal to  $\delta$  [6]. The space  $X$  is  $\delta$ -average rough if and only if the diameter of every convex combination of weak\* slices of  $B_{X^*}$  is greater than or equal to  $\delta$  [3].

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Banach spaces which are 2-average rough are exactly the *octahedral* ones (see [1], [3], and [4]). A weaker version of octahedrality was introduced in [5] and it was shown that a Banach space  $X$  is *weakly octahedral* if and only if the diameter of every non-empty relatively weak\* open subset of  $B_{X^*}$  is 2 [5, Theorem 2.8].

**Definition 1.2.** A Banach space  $X$  is said to be

- *octahedral* (see [4] and [5, Proposition 2.2]) if, whenever  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ , and  $\varepsilon > 0$ , there is a  $y \in S_X$  such that

$$\|x_i + y\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\};$$

- *weakly octahedral* (see [5, Proposition 2.6]) if, whenever  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ ,  $x^* \in B_{X^*}$ , and  $\varepsilon > 0$ , there is a  $y \in S_X$  such that

$$\|x_i + ty\| \geq (1 - \varepsilon)(|x^*(x_i)| + t) \quad \text{for all } i \in \{1, \dots, n\} \text{ and } t > 0.$$

Our note is motivated by the recent paper [2], where octahedrality of the space of bounded linear operators is studied. In Section 2, we give a unified improvement of the following Theorems 1.1 and 1.2 obtained in [2]. Moreover, we study their quantified versions in terms of roughness and average roughness (see Theorem 2.1).

**Theorem 1.1** ([2, Theorem 3.5]). *Let  $X$  and  $Y$  be Banach spaces, and let  $H$  be a closed subspace of  $\mathcal{L}(X, Y)$  containing the finite rank operators. If  $X^*$  and  $Y$  are octahedral, then  $H$  is octahedral.*

For the convenience of reference, let us point out a property for a Banach space  $X$  used as a hypothesis in Theorem 1.2:

(P) there is a  $u \in S_X$  such that the set

$$\{x^* \in B_{X^*} : x^*(u) = 1\}$$

is norming for  $X$  in the sense that, for every  $x \in S_X$  and every  $\varepsilon > 0$ , there is an  $x^* \in B_{X^*}$  such that

$$|x^*(x)| > 1 - \varepsilon \quad \text{and} \quad x^*(u) = 1.$$

**Theorem 1.2** ([2, Theorems 3.1 and 3.2]). *Let  $X$  and  $Y$  be Banach spaces, and let  $H$  be a closed subspace of  $\mathcal{L}(X, Y)$  containing the finite rank operators.*

- If  $X^*$  is octahedral and  $Y$  has property (P), then  $H$  is octahedral.*
- If  $X^*$  has property (P) and  $Y$  is octahedral, then  $H$  is octahedral.*

In Section 3, we first establish a quantitative version of the following Theorem 1.3 from [2] in terms of average roughness (see Theorem 3.1).

**Theorem 1.3** ([2, Proposition 3.9 and Corollary 3.10]). *Let  $X$  and  $Y$  be Banach spaces, and let  $H$  be a closed subspace of  $\mathcal{L}(X, Y)$  containing the finite rank operators.*

- (a) If  $H$  is octahedral and  $X^*$  is non-rough, then  $Y$  is octahedral.
- (b) If  $H$  is octahedral and  $Y$  is non-rough, then  $X^*$  is octahedral.

We then introduce a new notion of *weak  $\delta$ -average roughness* of a Banach space which corresponds to the property that the diameter of every non-empty relatively weak\* open subset of the dual unit ball is greater than or equal to  $\delta$  (see Theorem 3.3). Our main result in Section 3 is a quantitative version of Theorem 1.3 in terms of weak  $\delta$ -average roughness (see Theorem 3.5).

Let us fix some more notation. Let  $X$  and  $Y$  be Banach spaces. For  $x^* \in X^*$  and  $y \in Y$ , we denote by  $x^* \otimes y$  the operator in  $\mathcal{L}(X, Y)$  defined by  $(x^* \otimes y)(x) = x^*(x)y$ ,  $x \in X$ . For a subset  $A$  of  $X$ , its linear span and convex hull are denoted by  $\text{span}(A)$  and  $\text{conv}(A)$ , respectively.

## 2. SUFFICIENT CONDITIONS FOR ROUGHNESS IN SPACES OF OPERATORS

The main objective in this section is to relax the assumptions in Theorems 1.1 and 1.2. In order to do so, we introduce a new notion of *alternative octahedrality*, which in general is a weaker property than both octahedrality and property (P).

**Definition 2.1.** Let  $X$  be a Banach space. We say that  $X$  is *alternatively octahedral* if, whenever  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ , and  $\varepsilon > 0$ , there is a  $y \in S_X$  such that

$$\max\{\|x_i + y\|, \|x_i - y\|\} \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Note that the alternative octahedrality of  $X$  is equivalent to the following condition:

- whenever  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ , and  $\varepsilon > 0$ , there are  $y \in S_X$  and  $x_1^*, \dots, x_n^* \in S_{X^*}$  such that, for every  $i \in \{1, \dots, n\}$ ,

$$|x_i^*(x_i)| > 1 - \varepsilon \quad \text{and} \quad |x_i^*(y)| > 1 - \varepsilon.$$

Observe that both octahedrality and the property (P) above imply alternative octahedrality. On the other hand, for example,  $c_0$  is alternatively octahedral, but fails to be octahedral nor does it have property (P).

**Theorem 2.1.** Let  $X$  and  $Y$  be Banach spaces, let  $H$  be a closed subspace of  $\mathcal{L}(X, Y)$  containing the finite rank operators, and let  $\delta > 0$ .

- (a) If  $X^*$  is  $\delta$ -average rough and  $Y$  is alternatively octahedral, then  $H$  is  $\delta$ -average rough.
- (b) If  $X^*$  is alternatively octahedral and  $Y$  is  $\delta$ -average rough, then  $H$  is  $\delta$ -average rough.

Theorem 2.1 improves Theorems 1.1 and 1.2, because octahedrality is 2-average roughness. In particular, it shows that  $\mathcal{L}(c_0, c_0)$  is octahedral, while its octahedrality can not be deduced from Theorems 1.1 or 1.2. Also, Theorem 2.1 allows one to refine [2, Corollaries 3.3, 3.4, and 3.6].

*Proof of Theorem 2.1.* (a). Let  $n \in \mathbb{N}$ ,  $S_1, \dots, S_n \in S_H$ , and  $\varepsilon > 0$ . It suffices to find a  $T \in H$  with  $\|T\| = \varepsilon$  satisfying

$$\frac{1}{n} \sum_{i=1}^n (\|S_i + T\| + \|S_i - T\|) > (\delta - 5\varepsilon)\|T\| + 2.$$

Choose  $x_i \in S_X$  so that  $\|S_i x_i\| > 1 - \varepsilon^2$ . Since  $Y$  is alternatively octahedral, there are  $y \in S_Y$  and  $y_1^*, \dots, y_n^* \in S_{Y^*}$  such that, for every  $i \in \{1, \dots, n\}$ ,

$$|y_i^*(S_i x_i)| > 1 - \varepsilon^2 \quad \text{and} \quad y_i^*(y) > 1 - \varepsilon.$$

Since  $X^*$  is  $\delta$ -average rough, there is an  $x^* \in X^*$  with  $\|x^*\| = \varepsilon$  such that

$$\frac{1}{n} \sum_{i=1}^n (\|S_i^* y_i^* + x^*\| + \|S_i^* y_i^* - x^*\|) > (\delta - \varepsilon)\|x^*\| + \frac{2}{n} \sum_{i=1}^n \|S_i^* y_i^*\|.$$

Thus

$$\frac{1}{n} \sum_{i=1}^n (\|S_i^* y_i^* + x^*\| + \|S_i^* y_i^* - x^*\|) > (\delta - 3\varepsilon)\|x^*\| + 2.$$

Letting  $T := x^* \otimes y$ , one has  $\|T\| = \|x^*\| = \varepsilon$  and

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n (\|S_i + T\| + \|S_i - T\|) \\ & \geq \frac{1}{n} \sum_{i=1}^n (\|S_i^* y_i^* + T^* y_i^*\| + \|S_i^* y_i^* - T^* y_i^*\|) \\ & \geq \frac{1}{n} \sum_{i=1}^n (\|S_i^* y_i^* + x^*\| + \|S_i^* y_i^* - x^*\| - 2\|x^* - T^* y_i^*\|) \\ & = \frac{1}{n} \sum_{i=1}^n (\|S_i^* y_i^* + x^*\| + \|S_i^* y_i^* - x^*\| - 2(1 - y_i^*(y))\|x^*\|) \\ & > (\delta - 3\varepsilon)\|x^*\| + 2 - 2\varepsilon\|x^*\| \\ & = (\delta - 5\varepsilon)\|T\| + 2. \end{aligned}$$

(b). The proof is similar to that of (a). □

**Theorem 2.2.** *Let  $X$  and  $Y$  be Banach spaces, let  $H$  be a closed subspace of  $\mathcal{L}(X, Y)$  containing the finite rank operators, and let  $\delta > 0$ .*

- (a) *If  $X^*$  is  $\delta$ -rough, then  $H$  is  $\delta$ -rough.*
- (b) *If  $Y$  is  $\delta$ -rough, then  $H$  is  $\delta$ -rough.*

*Proof.* (a). We mimic the proof of Theorem 2.1(a) with  $n = 1$ . Let  $S \in S_H$  and  $\varepsilon > 0$ . It suffices to find a  $T \in H$  with  $\|T\| = \varepsilon$  satisfying

$$\|S + T\| + \|S - T\| > (\delta - 5\varepsilon)\|T\| + 2.$$

Let  $y^* \in S_{Y^*}$  be such that  $\|S^*y^*\| > 1 - \varepsilon^2$ . Let  $y \in S_Y$  be such that  $y^*(y) > 1 - \varepsilon$ . Since  $X^*$  is  $\delta$ -rough, there is an  $x^* \in X^*$  with  $\|x^*\| = \varepsilon$  such that

$$\|S^*y^* + x^*\| + \|S^*y^* - x^*\| > (\delta - \varepsilon)\|x^*\| + 2\|S^*y^*\|.$$

Thus

$$\|S^*y^* + x^*\| + \|S^*y^* - x^*\| > (\delta - 3\varepsilon)\|x^*\| + 2.$$

Letting  $T := x^* \otimes y$ , one has  $\|T\| = \|x^*\| = \varepsilon$  and

$$\begin{aligned} \|S + T\| + \|S - T\| &\geq \|S^*y^* + T^*y^*\| + \|S^*y^* - T^*y^*\| \\ &\geq \|S^*y^* + x^*\| + \|S^*y^* - x^*\| - 2\|x^* - T^*y^*\| \\ &= \|S^*y^* + x^*\| + \|S^*y^* - x^*\| - 2(1 - y^*(y))\|x^*\| \\ &> (\delta - 3\varepsilon)\|x^*\| + 2 - 2\varepsilon\|x^*\| \\ &= (\delta - 5\varepsilon)\|T\| + 2. \end{aligned}$$

(b). The proof is similar to that of (a).  $\square$

We do not know whether for the octahedrality of  $\mathcal{L}(X, Y)$  it is, in general, sufficient that only one of the spaces  $X^*$  or  $Y$  is octahedral without any additional assumptions (see also the discussion after Corollary 3.6 in [2]). We next show that, for  $1 < p < \infty$ , the space  $\mathcal{L}(c_0, \ell_p^2)$  is octahedral. Its octahedrality can not be deduced from Theorem 2.1, because  $\ell_p^2$  is not alternatively octahedral. The octahedrality of  $\mathcal{L}(c_0, \ell_1^n)$  and  $\mathcal{L}(c_0, \ell_\infty^n)$  is a direct consequence of Theorem 2.1, because  $\ell_1^n$  and  $\ell_\infty^n$  are both alternatively octahedral.

**Proposition 2.3.** *If  $1 < p < \infty$ , then  $\mathcal{L}(c_0, \ell_p^2)$  is octahedral.*

Our proof of Proposition 2.3 uses the following lemma.

**Lemma 2.4.** *Let  $1 < p < \infty$ , let  $n \in \mathbb{N}$ , and let  $a_1 = (\alpha_1, \beta_1), \dots, a_n = (\alpha_n, \beta_n) \in S_{\ell_p^2}$  be such that  $\alpha_1, \dots, \alpha_n \geq 0$  and  $\beta_1 \geq \dots \geq \beta_n$ . Then*

$$\theta_1 \cdot \frac{a_1 + a_n}{2} + \theta_2 \cdot \frac{a_2 - a_1}{2} + \dots + \theta_n \cdot \frac{a_n - a_{n-1}}{2} \in B_{\ell_p^2}$$

for all  $\theta_1, \dots, \theta_n \in \{-1, 1\}$ .

*Proof.* Let  $\theta_1, \dots, \theta_n \in \{-1, 1\}$ . Put

$$x := \theta_1 \cdot \frac{a_1 + a_n}{2} + \theta_2 \cdot \frac{a_2 - a_1}{2} + \dots + \theta_n \cdot \frac{a_n - a_{n-1}}{2}.$$

We will show that  $x \in B_{\ell_p^2}$ . Without loss of generality we may assume that  $\theta_1 = 1$ . Since

$$\frac{a_n}{2} = \frac{a_1}{2} + \frac{a_2 - a_1}{2} + \dots + \frac{a_n - a_{n-1}}{2},$$

we have that

$$x = a_1 + \frac{a_2 - a_1}{2} + \cdots + \frac{a_n - a_{n-1}}{2} + \\ + \theta_2 \cdot \frac{a_2 - a_1}{2} + \cdots + \theta_n \cdot \frac{a_n - a_{n-1}}{2}.$$

Hence there is an odd number of increasing indices  $k_1, \dots, k_{2l+1}$  such that  $x$  is representable as

$$(2.1) \quad x = a_{k_1} - a_{k_2} + a_{k_3} - \cdots - a_{k_{2l}} + a_{k_{2l+1}}.$$

To show that  $x \in B_{\ell_p^2}$ , we use the following geometric properties of  $\ell_p^2$ .

**Fact.** For  $a, b \in S_{\ell_p^2}$ , let  $\Theta_{a,b} := B_{\ell_p^2} \cap (B_{\ell_p^2} + (a + b))$ .

- (a) If  $a, b \in S_{\ell_p^2}$  and  $y \in \Theta_{a,b}$ , then  $a - y + b \in \Theta_{a,b}$ .
- (b) If  $a, b$ , and  $c$  are pairwise different elements of  $S_{\ell_p^2}$  and  $b \in \Theta_{a,c}$ , then  $\Theta_{a,b} \subset \Theta_{a,c}$ .

Since  $a_{k_{l+1}} \in \Theta_{a_{k_l}, a_{k_{l+2}}}$ , we have that  $z := a_{k_l} - a_{k_{l+1}} + a_{k_{l+2}} \in \Theta_{a_{k_l}, a_{k_{l+2}}}$  by part (a) of Fact. We can write the middle part of the right hand side of (2.1) as

$$\cdots a_{k_{l-1}} - (a_{k_l} - a_{k_{l+1}} + a_{k_{l+2}}) + a_{k_{l+3}} \cdots$$

By part (b) of Fact,  $z \in \Theta_{a_{k_l}, a_{k_{l+2}}} \subset \Theta_{a_{k_{l-1}}, a_{k_{l+3}}}$ . Applying part (a) of Fact we have that  $a_{k_{l-1}} - z + a_{k_{l+3}} \in \Theta_{a_{k_{l-1}}, a_{k_{l+3}}}$ . Continuing in this way, we will finally have  $x \in \Theta_{a_{k_1}, a_{k_{2l+1}}} \subset B_{\ell_p^2}$ .  $\square$

*Proof of Proposition 2.3.* Let  $n \in \mathbb{N}$ ,  $S_1, \dots, S_n \in S_{\mathcal{L}(c_0, \ell_p^2)}$ , and  $\varepsilon \in (0, 1)$ . It suffices to show that there is a  $T \in S_{\mathcal{L}(c_0, \ell_p^2)}$  such that

$$\|S_i + T\| \geq 2 - 3\varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Choose  $x_i \in S_{c_0}$  such that  $\|S_i x_i\| \geq 1 - \varepsilon$ . Without loss of generality we may assume that  $x_1, \dots, x_n$  are finitely supported, that is, there is a  $N_1 \in \mathbb{N}$  such that  $x_1, \dots, x_n \in \text{span}\{e_1, \dots, e_{N_1}\}$ .

Since  $S_1, \dots, S_n$  are finite rank operators and  $(e_k)$  is a weakly null sequence in  $c_0$ , there is a  $N_2 \in \mathbb{N}$  such that  $\|S_i e_k\| \leq \varepsilon/n$  for all  $i \in \{1, \dots, n\}$  and  $k \geq N_2$ . Take  $N = \max\{N_1, N_2\}$ .

For all  $i \in \{1, \dots, n\}$ , put  $a_i := S_i x_i / \|S_i x_i\|$ . By reordering  $a_1, \dots, a_n$  and by replacing  $a_i$  with  $-a_i$  if necessary, we may assume that  $a_1, \dots, a_n$  satisfy the assumptions of Lemma 2.4.

Define  $T: c_0 \rightarrow \ell_p^2$  by

$$Te_{N+1} = \frac{a_1 + a_n}{2}, \quad Te_{N+2} = \frac{a_2 - a_1}{2}, \quad \dots, \quad Te_{N+n} = \frac{a_n - a_{n-1}}{2},$$

and  $Te_k = 0$ , if  $k \in \mathbb{N} \setminus \{N+1, \dots, N+n\}$ .

By Lemma 2.4,  $\|T\| \leq 1$ . On the other hand,  $\|T\| \geq 1$ , because  $T(e_{N+1} + \cdots + e_{N+n}) = a_n$ . Thus  $\|T\| = 1$ .

Fix  $i \in \{1, \dots, n\}$ . Choose  $\theta_1, \dots, \theta_n \in \{-1, 1\}$  so that

$$\theta_1 \cdot \frac{a_1 + a_n}{2} + \theta_2 \cdot \frac{a_2 - a_1}{2} + \dots + \theta_n \cdot \frac{a_n - a_{n-1}}{2} = a_i.$$

Let  $y_i := \theta_1 e_{N+1} + \dots + \theta_n e_{N+n}$ . Since  $Ty_i = a_i = S_i x_i / \|S_i x_i\|$ ,  $Tx_i = 0$ , and  $x_i + y_i \in S_{c_0}$ , we get

$$\begin{aligned} \|S_i + T\| &\geq \|(S_i + T)(x_i + y_i)\| \\ &= \|S_i x_i + S_i y_i + Ty_i\| \\ &\geq \|S_i x_i + Ty_i\| - \|S_i y_i\| \\ &\geq 2\|S_i x_i\| - \varepsilon \geq 2 - 3\varepsilon. \end{aligned}$$

□

### 3. NECESSARY CONDITIONS FOR ROUGHNESS IN SPACES OF OPERATORS

In this section, we first prove a quantitative version of Theorem 1.3 in terms of roughness. Our main result is a quantitative version of Theorem 1.3 for weakly octahedral Banach spaces.

Recall that a Banach space is non-rough if and only if its dual unit ball has weak\* slices of arbitrarily small diameter [6, Proposition 1].

**Theorem 3.1.** *Let  $X$  and  $Y$  be Banach spaces, let  $H$  be a closed subspace of  $\mathcal{L}(X, Y)$  containing the finite rank operators, and let  $\delta > 0$ .*

- (1) *Let  $H$  be  $\delta$ -average rough.*
  - (a) *If  $X^*$  is non-rough, then  $Y$  is  $\delta$ -average rough.*
  - (b) *If  $Y$  is non-rough, then  $X^*$  is  $\delta$ -average rough.*
- (2) *Let  $H$  be  $\delta$ -rough.*
  - (a) *If  $X^*$  is non-rough, then  $Y$  is  $\delta$ -rough.*
  - (b) *If  $Y$  is non-rough, then  $X^*$  is  $\delta$ -rough.*

*Proof.* (1a). Let  $n \in \mathbb{N}$ ,  $y_1, \dots, y_n \in S_Y$ , and  $\varepsilon \in (0, 1/3)$ . For the  $\delta$ -average roughness of  $Y$ , it suffices to find a  $z \in Y$  with  $\|z\| < \varepsilon$  such that

$$\frac{1}{n} \sum_{i=1}^n (\|y_i + z\| + \|y_i - z\|) > 2 + (\delta - 5\varepsilon)\|z\|.$$

Since  $X^*$  is non-rough, there are  $x^* \in S_{X^*}$  and  $\alpha \in (0, 3\varepsilon)$  such that, for the slice  $S(x^*, \alpha) := \{x \in B_X : x^*(x) > 1 - \alpha\}$ , one has  $\text{diam } S(x^*, \alpha) < \varepsilon$ .

Let  $S_i := x^* \otimes y_i$  for every  $i \in \{1, \dots, n\}$ . Since  $H$  is  $\delta$ -average rough, there is a  $T \in H$  with  $\|T\| < \frac{\alpha}{3}$  such that

$$\frac{1}{n} \sum_{i=1}^n (\|S_i + T\| + \|S_i - T\|) > 2 + (\delta - \varepsilon)\|T\|.$$

For  $j \in \{1, 2\}$ , choosing  $x_{i,j} \in S_X$  and  $y_{i,j}^* \in S_{Y^*}$  so that

$$\begin{aligned} x^*(x_{i,j}) y_{i,j}^*(y_i) + (-1)^{j+1} y_{i,j}^*(Tx_{i,j}) &= y_{i,j}^*(S_i x_{i,j} + (-1)^{j+1} Tx_{i,j}) \\ &\geq \|S_i + (-1)^{j+1} T\| - \varepsilon \|T\|, \end{aligned}$$

one may assume that both  $x^*(x_{i,j}) > 0$  and  $y_{i,j}^*(y_i) > 0$ , and thus  $x_{i,j} \in S(x^*, \alpha)$ , because

$$\begin{aligned} x^*(x_{i,j}) &\geq \|S_i + (-1)^{j+1} T\| - \varepsilon \|T\| - |y_{i,j}^*(Tx_{i,j})| \\ &\geq \|S_i\| - \|T\| - \varepsilon \|T\| - \|T\| \geq \|S_i\| - 3\|T\| > 1 - \alpha. \end{aligned}$$

Letting  $z := Tx_{1,1}$ , we have that  $\|z\| \leq \|T\| < \frac{\alpha}{3} < \varepsilon$  and, for every  $i \in \{1, \dots, n\}$ ,

$$\begin{aligned} \|y_i + z\| + \|y_i - z\| &= \|y_i + Tx_{1,1}\| + \|y_i - Tx_{1,1}\| \\ &\geq \|y_i + Tx_{i,1}\| + \|y_i - Tx_{i,2}\| \\ &\quad - \|Tx_{1,1} - Tx_{i,1}\| - \|Tx_{1,1} - Tx_{i,2}\| \\ &\geq y_{i,1}^*(y_i + Tx_{i,1}) + y_{i,2}^*(y_i - Tx_{i,2}) \\ &\quad - \|T\| \|x_{1,1} - x_{i,1}\| - \|T\| \|x_{1,1} - x_{i,2}\| \\ &\geq x^*(x_{i,1}) y_{i,1}^*(y_i) + y_{i,1}^*(Tx_{i,1}) \\ &\quad + x^*(x_{i,2}) y_{i,2}^*(y_i) - y_{i,2}^*(Tx_{i,2}) - 2\varepsilon \|T\| \\ &\geq \|S_i + T\| - \varepsilon \|T\| + \|S_i - T\| - \varepsilon \|T\| - 2\varepsilon \|T\| \\ &= \|S_i + T\| + \|S_i - T\| - 4\varepsilon \|T\|, \end{aligned}$$

and thus

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n (\|y_i + z\| + \|y_i - z\|) &\geq \frac{1}{n} \sum_{i=1}^n (\|S_i + T\| + \|S_i - T\|) - 4\varepsilon \|T\| \\ &> 2 + (\delta - 5\varepsilon) \|T\| \geq 2 + (\delta - 5\varepsilon) \|z\|. \end{aligned}$$

(1b). The proof is similar to that of (1a).

(2). The proof is exactly that of (1) with  $n = 1$ .  $\square$

Note that Theorem 3.1(1) with  $\delta = 2$  is exactly Theorem 1.3. Combining Theorems 2.1 and 3.1 improves [2, Corollary 3.11] as follows.

**Corollary 3.2.** *Let  $X$  and  $Y$  be Banach spaces, and let  $H$  be a closed subspace of  $\mathcal{L}(X, Y)$  containing the finite rank operators. Suppose that  $X^*$  is non-rough and alternatively octahedral. The following assertions are equivalent:*

- (i)  $H$  is octahedral;
- (ii)  $Y$  is octahedral.



The following definition is motivated by the known dual characterizations of (average) roughness in terms of the diameter of (convex combinations of) weak\* slices (see Introduction).

**Definition 3.1.** Let  $X$  be a Banach space and  $\delta > 0$ . We say that the space  $X$  is *weakly  $\delta$ -average rough* if every non-empty relatively weak\* open subset of  $B_{X^*}$  has diameter greater than or equal to  $\delta$ .

**Theorem 3.3.** Let  $\delta > 0$ . The following assertions are equivalent:

- (i)  $X$  is weakly  $\delta$ -average rough;
- (ii) whenever  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ ,  $x^* \in B_{X^*}$ , and  $\varepsilon, t_0 > 0$ , there is a  $y \in S_X$  such that,

$$\inf_{\substack{i \in \{1, \dots, n\} \\ t \geq t_0}} \frac{\|x_i + ty\| - x^*(x_i)}{t} - \sup_{\substack{i \in \{1, \dots, n\} \\ t \geq t_0}} \frac{\|x_i - ty\| - x^*(x_i)}{-t} > \delta - \varepsilon;$$

- (iii) whenever  $E$  is a finite-dimensional subspace of  $X$ ,  $x^* \in B_{X^*}$ , and  $\varepsilon, t_0 > 0$ , there is a  $y \in S_X$  such that

$$(3.1) \quad \inf_{\substack{x \in S_E \\ t \geq t_0}} \frac{\|x + ty\| - x^*(x)}{t} - \sup_{\substack{x \in S_E \\ t \geq t_0}} \frac{\|x - ty\| - x^*(x)}{-t} > \delta - \varepsilon;$$

- (iv) whenever  $E$  is a finite-dimensional subspace of  $X$ ,  $x^* \in B_{X^*}$ , and  $\varepsilon > 0$ , there are  $x_1^*, x_2^* \in X^*$  with  $\|x_1^*\|, \|x_2^*\| \leq 1 + \varepsilon$ , and  $y \in S_X$  satisfying

$$x_1^*|_E = x_2^*|_E = x^*|_E \quad \text{and} \quad x_1^*(y) - x_2^*(y) > \delta - \varepsilon;$$

- (v) whenever  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ ,  $x^* \in S_{X^*}$ , and  $\varepsilon > 0$ , there are  $x_1^*, x_2^* \in B_{X^*}$  and  $y \in B_X$  satisfying

$$|x_j^*(x_i) - x^*(x_i)| < \varepsilon \quad \text{for all } i \in \{1, \dots, n\} \text{ and } j \in \{1, 2\},$$

and  $x_1^*(y) - x_2^*(y) > \delta - \varepsilon$ .

The proof of the implication (iii) $\Rightarrow$ (iv) of Theorem 3.3 relies on the following lemma which explains the role of the inf and sup in the condition (iii).

**Lemma 3.4.** Let  $E$  be a finite-dimensional subspace of  $X$ ,  $y \in S_X \setminus E$ ,  $x^* \in B_{X^*}$ ,  $t_0 \in (0, 1)$ , and  $\gamma$  is such that

$$a := \sup_{\substack{x \in S_E \\ t \geq t_0}} \frac{\|x - ty\| - x^*(x)}{-t} \leq \gamma \leq \inf_{\substack{x \in S_E \\ t \geq t_0}} \frac{\|x + ty\| - x^*(x)}{t} =: b.$$

Then, for the functional

$$g: \text{span}(E \cup \{y\}) \ni x + ty \mapsto x^*(x) + t\gamma, \quad x \in E, t \in \mathbb{R},$$

one has  $\|g\| \leq \frac{1+t_0}{1-t_0}$ .

*Proof.* First observe that  $|\gamma| \leq 1$ , because, letting  $t \rightarrow \infty$ , one obtains  $a \geq -1$  and  $b \leq 1$ .

In order that  $\|g\| \leq \frac{1+t_0}{1-t_0}$ , it suffices to show that, letting  $x \in S_E$  and  $t > 0$ , one has

$$|x^*(x) + t\gamma| \leq \frac{1+t_0}{1-t_0} \|x + ty\|.$$

For  $0 < t \leq t_0$ , one has

$$\begin{aligned} |x^*(x) + t\gamma| &\leq 1 + t = \frac{1+t}{1-t} (\|x\| - t\|y\|) \leq \frac{1+t}{1-t} \|x + ty\| \\ &\leq \frac{1+t_0}{1-t_0} \|x + ty\|. \end{aligned}$$

For  $t \geq t_0$ , one has, in fact, a stronger inequality:

$$|x^*(x) + t\gamma| \leq \|x + ty\|.$$

Indeed, the latter condition is equivalent to

$$\frac{\| -x - ty \| - x^*(-x)}{-t} \leq \gamma \leq \frac{\|x + ty\| - x^*(x)}{t}$$

which holds because  $\gamma \in [a, b]$ .  $\square$

*Proof of Theorem 3.3.* (i) $\Leftrightarrow$ (v) and (iv) $\Rightarrow$ (v) are obvious.

(v) $\Rightarrow$ (ii). Assume that (v) holds. Let  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ ,  $x^* \in B_{X^*}$ , and  $\varepsilon, t_0 > 0$ . By (v), there are  $u^*, v^* \in B_{X^*}$  and  $y \in S_X$  such that, for every  $i \in \{1, \dots, n\}$ ,

$$|u^*(x_i) - x^*(x_i)| < \varepsilon t_0 \quad \text{and} \quad |v^*(x_i) - x^*(x_i)| < \varepsilon t_0,$$

and

$$v^*(y) - u^*(y) > \delta - \varepsilon.$$

Now let  $i \in \{1, \dots, n\}$  and  $t \geq t_0$  be arbitrary. Then

$$\begin{aligned} v^*(y) &= \frac{v^*(x_i + ty) - v^*(x_i)}{t} \leq \frac{\|x_i + ty\| - v^*(x_i)}{t} \\ &< \frac{\|x_i + ty\| - x^*(x_i) + \varepsilon t_0}{t} \leq \frac{\|x_i + ty\| - x^*(x_i)}{t} + \varepsilon \end{aligned}$$

and

$$\begin{aligned} -u^*(y) &= \frac{u^*(x_i - ty) - u^*(x_i)}{t} \leq \frac{\|x_i - ty\| - u^*(x_i)}{t} \\ &< \frac{\|x_i - ty\| - x^*(x_i) + \varepsilon t_0}{t} \leq \frac{\|x_i - ty\| - x^*(x_i)}{t} + \varepsilon. \end{aligned}$$

It follows that

$$\begin{aligned} &\inf_{\substack{i \in \{1, \dots, n\} \\ t \geq t_0}} \frac{\|x_i + ty\| - x^*(x_i)}{t} - \sup_{\substack{i \in \{1, \dots, n\} \\ t \geq t_0}} \frac{\|x_i - ty\| - x^*(x_i)}{-t} \\ &\geq v^*(y) - \varepsilon - u^*(y) - \varepsilon > \delta - 3\varepsilon. \end{aligned}$$

(ii) $\Rightarrow$ (iii). Let  $E$  be a finite-dimensional subspace of  $X$ ,  $x^* \in B_{X^*}$ , and  $\varepsilon, t_0 > 0$ . It suffices to observe that, letting  $y \in S_X$  be produced by (ii) where  $\{x_1, \dots, x_n\} \subset S_X$  is an  $\varepsilon t_0$ -net for  $S_E$ , the difference of the inf and sup in (iii) is greater than  $\delta - 5\varepsilon$ .

(iii) $\Rightarrow$ (iv). Assume that (iii) holds. Let  $E$  be a finite-dimensional subspace of  $X$ ,  $x^* \in B_{X^*}$ , and  $\varepsilon > 0$ . Choosing  $t_0 \in (0, 1)$  so that  $\frac{1+t_0}{1-t_0} < 1 + \varepsilon$ , let  $y \in S_X$  satisfy (3.1). Define  $g_1, g_2 \in (\text{span}(E \cup \{y\}))^*$  in the same manner as  $g$  in Lemma 3.4 where, respectively,  $\gamma = b$  and  $\gamma = a$ . One may let the desired  $x_1^*$  and  $x_2^*$  be any norm preserving extensions to  $X$  of  $g_1$  and  $g_2$ , respectively.  $\square$

**Theorem 3.5.** *Let  $X$  and  $Y$  be Banach spaces, let  $H$  be a closed subspace of  $\mathcal{L}(X, Y)$  containing the finite rank operators, and let  $\delta > 0$ . Suppose that  $H$  is weakly  $\delta$ -average rough.*

- (a) *If  $X^*$  is non-rough, then  $Y$  is weakly  $\delta$ -average rough.*
- (b) *If  $Y$  is non-rough, then  $X^*$  is weakly  $\delta$ -average rough.*

*Proof.* (a). Let  $n \in \mathbb{N}$ ,  $y_1, \dots, y_n \in S_Y$ ,  $y^* \in S_{Y^*}$ , and  $\varepsilon \in (0, 1)$ . For the weak  $\delta$ -average roughness of  $Y$ , by Theorem 3.3, it suffices to find  $y_1^*, y_2^* \in B_{Y^*}$  and  $y \in B_Y$  such that

$$|y_j^*(y_i) - y^*(y_i)| < 6\varepsilon \quad \text{for all } i \in \{1, \dots, n\} \text{ and } j \in \{1, 2\},$$

and

$$y_1^*(y) - y_2^*(y) > \delta - 13\varepsilon.$$

Since  $X^*$  is non-rough, there are  $x^* \in S_{X^*}$  and  $\alpha \in (0, \varepsilon)$  such that, for the slice  $S(x^*, \alpha) := \{x \in B_X : x^*(x) > 1 - \alpha\}$ , one has  $\text{diam } S(x^*, \alpha) < \varepsilon$ .

Choose  $x^{**} \in S_{X^{**}}$  and  $y_0 \in S_Y$  so that  $x^{**}(x^*) = 1$  and  $y^*(y_0) > 1 - \alpha^2$ , and put  $S_i := x^* \otimes y_i \in H$  for every  $i \in \{0, 1, \dots, n\}$ , and  $\phi := x^{**} \otimes y^* \in H^*$ , where

$$(x^{**} \otimes y^*)(S) = x^{**}(S^* y^*), \quad S \in H.$$

Since  $H$  is weakly  $\delta$ -average rough, by Theorem 3.3, there are  $\phi_1, \phi_2 \in H^*$  with  $\|\phi_1\|, \|\phi_2\| < 1 + \alpha^2$ , and  $T \in S_H$  such that

$$\phi_1(S_i) = \phi_2(S_i) = \phi(S_i) = y^*(y_i) \quad \text{for all } i \in \{0, 1, \dots, n\},$$

and

$$\phi_1(T) - \phi_2(T) > \delta - \varepsilon.$$

Denote by  $B_X \otimes B_{Y^*}$  the set of functionals in  $H^*$  of the form  $x \otimes y^*$ , where  $x \in B_X$  and  $y^* \in Y^*$ . Since, for every  $S \in H$ , there is some  $f$  in the weak\* closure of  $B_X \otimes B_{Y^*}$  in  $H^*$  such that  $f(S) = \|S\|$ , by the Hahn–Banach separation theorem, it quickly follows that  $\text{conv}(B_X \otimes B_{Y^*})$  is weak\* dense in  $B_{H^*}$ . Thus, for all  $j \in \{1, 2\}$ , observing that

$\left\| \frac{1}{\|\phi_j\|} \phi_j - \phi_j \right\| < \alpha^2$ , there are

$$\psi_j := \sum_{k=1}^{m_j} \lambda_{j,k} x_{j,k} \otimes y_{j,k}^* \in \text{conv}(B_X \otimes B_{Y^*})$$

such that  $|\psi_j(T) - \phi_j(T)| < \alpha^2$  and, for all  $i \in \{0, 1, \dots, n\}$ ,

$$\left| \sum_{k=1}^{m_j} \lambda_{j,k} x^*(x_{j,k}) y_{j,k}^*(y_i) - y^*(y_i) \right| = |\psi_j(S_i) - \phi_j(S_i)| < \alpha^2.$$

Letting

$$K_j := \{k \in \{1, \dots, m_j\} : x^*(x_{j,k}) > 1 - \alpha\} \quad \text{and} \quad \lambda_j := \sum_{k \notin K_j} \lambda_{j,k},$$

one has

$$\begin{aligned} 1 - 2\alpha^2 &< y^*(y_0) - \alpha^2 < \sum_{k=1}^{m_j} \lambda_{j,k} x^*(x_{j,k}) y_{j,k}^*(y_0) \\ &= \sum_{k \notin K_j} \lambda_{j,k} x^*(x_{j,k}) y_{j,k}^*(y_0) + \sum_{k \in K_j} \lambda_{j,k} x^*(x_{j,k}) y_{j,k}^*(y_0) \\ &\leq (1 - \alpha) \sum_{k \notin K_j} \lambda_{j,k} + \sum_{k \in K_j} \lambda_{j,k} = (1 - \alpha) \lambda_j + 1 - \lambda_j \\ &= 1 - \alpha \lambda_j \end{aligned}$$

whence  $\lambda_j < 2\alpha$ . Now, putting  $y_j^* := \sum_{k=1}^{m_j} \lambda_{j,k} y_{j,k}^*$ , one has

$$\begin{aligned} |y_j^*(y_i) - y^*(y_i)| &= \left| \sum_{k=1}^{m_j} \lambda_{j,k} y_{j,k}^*(y_i) - y^*(y_i) \right| \\ &\leq \left| \sum_{k=1}^{m_j} \lambda_{j,k} x^*(x_{j,k}) y_{j,k}^*(y_i) - y^*(y_i) \right| \\ &\quad + \sum_{k \notin K_j} \lambda_{j,k} |1 - x^*(x_{j,k})| |y_{j,k}^*(y_i)| \\ &\quad + \sum_{k \in K_j} \lambda_{j,k} |1 - x^*(x_{j,k})| |y_{j,k}^*(y_i)| \\ &< \alpha^2 + 2\lambda_j + \alpha < 6\alpha < 6\varepsilon. \end{aligned}$$

Letting  $x \in S(x^*, \alpha)$  be arbitrary, one has

$$\|x - x_{j,k}\| < \varepsilon \quad \text{for all } k \in K_j,$$

thus

$$\begin{aligned}
|y_j^*(Tx) - \psi_j(T)| &= \left| \sum_{k=1}^{m_j} \lambda_{j,k} (T^* y_{j,k}^*)(x - x_{j,k}) \right| \\
&\leq \sum_{k \notin K_j} \lambda_{j,k} \|x - x_{j,k}\| + \sum_{k \in K_j} \lambda_{j,k} \|x - x_{j,k}\| \\
&\leq 2\lambda_j + \varepsilon < 4\alpha + \varepsilon < 5\varepsilon,
\end{aligned}$$

and it follows that

$$\begin{aligned}
y_1^*(Tx) - y_2^*(Tx) &= y_1^*(Tx) - \psi_1(T) + \psi_1(T) - \phi_1(T) + \phi_1(T) - \phi_2(T) \\
&\quad + \phi_2(T) - \psi_2(T) + \psi_2(T) - y_2^*(Tx) \\
&\geq \phi_1(T) - \phi_2(T) \\
&\quad - |y_1^*(Tx) - \psi_1(T)| - |\psi_1(T) - \phi_1(T)| \\
&\quad - |\phi_2(T) - \psi_2(T)| - |\psi_2(T) - y_2^*(Tx)| \\
&> \delta - \varepsilon - 5\varepsilon - \alpha^2 - \alpha^2 - 5\varepsilon > \delta - 13\varepsilon.
\end{aligned}$$

(b). The proof is similar to that of (a).  $\square$

Taking  $\delta = 2$  in Theorem 3.5 we get the following corollary.

**Corollary 3.6.** *Let  $X$  and  $Y$  be Banach spaces, and let  $H$  be a closed subspace of  $\mathcal{L}(X, Y)$  containing the finite rank operators. Suppose that  $H$  is weakly octahedral.*

- (a) *If  $X^*$  is non-rough, then  $Y$  is weakly octahedral.*
- (b) *If  $Y$  is non-rough, then  $X^*$  is weakly octahedral.*

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